Rational Function Multiplicative Coefficients

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1 Rational generating functions of multiplicative sequences

Any numerical sequence has an associated generating function (GF). For example, the Fibonacci sequence is associated with GF $x/(1-x-x^2)$, a rational function of x. Consider a multiplicative sequence. That is, $a(1) = 1$ and $a(nm) = a(n)a(m)$ for all positive integers n and m relatively prime to each other. Can its GF $f(x) = a(1)x + a(2)x^2 + a(3)x^3 + ...$ ever be rational? The answer is yes if $f(x) = x/(1-x)$ and $a(n) = 1$ if $n > 0$. This is the simplest example where $a(n)$ is non-zero for all $n > 0$. Another is $f(x) = x/(1-x^2)$ and $a(n) = 1$ if $n > 0$ is odd and $a(n) = 0$ otherwise. Now consider the rational function and its power series expansion

$$f(x) = x(1-x)^{e_1}(1-x^2)^{e_2} = x - e_1x^2 + ((e_1^2 - e_1)/2 - e_2)x^3 + ...$$

where e_1 and e_2 are integers. A search finds that $f(x)$ is the GF of a multiplicative sequence for 11 pairs of integers $[e_1, e_2]$ as follows:

$$[-4, 1], [-2, 0], [-1, 0], [-1, 1], [0, -1], [0, 0], [0, 1], [1, -1], [1, 0], [2, -2], [4, -3].$$

The multiplicative integer sequences for these pairs are of a simple form. Some algebra is enough to prove that this list is complete. Allowing more factors in $f(x)$ increases the difficulty of search and algebraic proof.

2 Conjecture 1

Conjecture 1: there is a finite set of rational functions of the form

$$f(x) = x(1-x)^{e_1}(1-x^2)^{e_2}...(1-x^n)^{e_n}$$

for some integers $e_1, ..., e_n$ which are the GF for multiplicative integer sequences provided we exclude some infinite families which are predictable. One example infinite family is

$$f(x) = x(1-x^{n-1}) = x - x^n$$

where $n > 1$. Also, if and only if $n = p^k, n > 1$ and p is prime then

$$f(x) = x(1-x)^{-1}(1-x^{n-1})(1-x^n)^{-1}$$

is multiplicative. Note that $f(x)$ is in the set when $-f(-x)$ is since $(1 + x) = (1-x^2)/(1-x)$ and so on.

3 Homogeneous generalization of multiplicative sequences

Now assume $a(0)$ and $a(1)$ are nonzero. For example, consider the sequence $a(n)$ with its GF

$$g(x) = (1-x)^2/(1-x^2) = 1 - 2x + 2x^2 - 2x^3 + \dots.$$

Then $a(1)a(nm) = a(n)a(m)$ for all positive integers n and m relatively prime to each other. This is a homogeneous generalization of multiplicative sequences. As in the first section, but without a factor of x, consider

$$g(x) = (1-x)^{e_1}(1-x^2)^{e_2} = 1 - e_1x + ((e_1^2 - e_1)/2 - e_2)x^2 + \dots$$

where e_1 and e_2 are integers. A search finds that $g(x)$ is the GF of a homogeneous multiplicative sequence for 10 pairs of integers $[e_1, e_2]$ as follows:

$$[-4,2], [-2,1], [-2,2], [-1,0], [-1,1], [1,-1], [1,0], [2,-1], [2,0], [4,-2].$$

Again, algebra is enough to prove the list is complete.

4 Conjecture 2

Conjecture 2: there is a finite set of rational functions of the form

$$g(x) = (1-x)^{e_1}(1-x^2)^{e_2}\dots(1-x^n)^{e_n}$$

for some integers $e_1, \dots, e_n$ which are the GF for homogeneous multiplicative integer sequences provided we exclude some infinite families which are predictable. For example,

$$g(x) = (1-x)^{-1}(1-x^n)^{-1}(1-x^{n+1})$$

is homogeneous multiplicative if and only if $n = p^k, n > 1$ and p is prime. Note that $g(x)$ is in the set when $g(-x)$ is.

5 Further Work

The rational functions in the two conjectures have applications related to Ramanujan's Lambert series. A study of rational functions with poles only at roots of unity appeared in 2003 by Juan B. Gil and Sinai Robins who defined a Hecke operator on power series. Kyoji Saito studied cyclotomic functions related to eta-products in 2001. Rational functions of a simple form having multiplicative coefficients is related to a paper on Multiplicative η- Quotients by Yves Martin in 1996.